

Non-perturbative Aspect of QFT

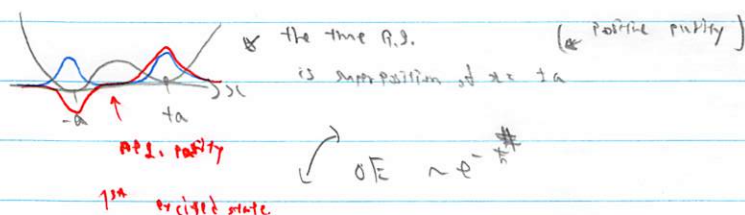
- Semi-classical approach via compactifications. 1/2 of 1/2 of 1/2.

Outline

- Dilute - Instanton gas in QM
- 2d U(1) gauge theories
- 4d YM / eCD on $R^2 \times T^2$ twisted

1. Dilute - Instanton gas approximation in QM.

$$\hat{H} = -\frac{1}{2}\hbar^2 + V(x)$$



Let's derive this fact using path integral

$$\begin{aligned} \langle x_f | e^{-\frac{\hat{H}}{\hbar} \tau} | x_i \rangle \\ = \int \mathcal{D}x \, e^{-\frac{1}{\hbar} \int_0^\tau dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right)} \end{aligned}$$

↓ Wick rotation $e^{-\frac{\hat{H}}{\hbar} \tau} \Rightarrow e^{-\hat{H} \tau}$

$$\begin{aligned} \langle x_f | e^{-\hat{H} \tau/\hbar} | x_i \rangle \\ = \int_{x(0)=x_i}^{x(\tau)=x_f} \mathcal{D}x \, e^{-\frac{1}{\hbar} \int_0^\tau dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right)} \end{aligned}$$

Thm $\hat{H}$ of $\hat{H} = -\frac{1}{2}\hbar^2 + V(x)$ is unique (where $V(x)$ is a positive function.)

↓ eigenstate

We can take $V(x)$ to be a positive function.

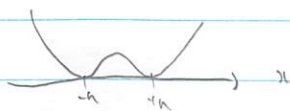
(Proof) From Euclidean path integral

$e^{-\hat{H} \tau/\hbar}$ is a matrix with positive entries

$$= \begin{pmatrix} \langle \phi^0 | \phi^0 \rangle & \dots \\ \vdots & \ddots \end{pmatrix} \quad (\text{in coord. basis})$$

= Perron - Frobenius thm tells the result.

$$V(x) = \frac{1}{4} (x^2 - a^2)^2$$



$$\begin{pmatrix} \langle a | e^{-\hat{H}T} | a \rangle & \langle a | e^{-\hat{H}T} | -a \rangle \\ \langle -a | e^{-\hat{H}T} | a \rangle & \langle -a | e^{-\hat{H}T} | -a \rangle \end{pmatrix} \approx \frac{1}{\hbar \omega} \ll \ll \left(\text{non-instantaneous} \right)^{-1}$$

$$\langle a | e^{-\hat{H}T} | a \rangle = \langle -a | e^{-\hat{H}T} | -a \rangle$$

$$= \int_{x(0)=-a}^{x(T)=a} \mathcal{D}x \cdot e^{-\frac{1}{\hbar} \int_0^T dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right)}$$

$$\approx 1 \quad (x(t) = +a \text{ for } t < 0)$$

$$\langle -a | e^{-\hat{H}T} | a \rangle = \langle a | e^{-\hat{H}T} | -a \rangle$$

$$= \int_{x(0)=a}^{x(T)=-a} \mathcal{D}x \cdot e^{-\frac{1}{\hbar} \int_0^T dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right)}$$

To evaluate this path int.,

we want to solve

$$\text{EoM of } \phi \quad \text{w/} \quad \begin{cases} x(0) = -a \\ x(T) = +a \end{cases}$$

There is a clever trick for this purpose:

Bogomolnyi - Prasad - Sommerfield (BPS) eq.

$$SE(x) = \int_0^T dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right)$$

$$= \int_0^T dt \left(\frac{1}{2} (\dot{x} \mp \sqrt{2V(x)})^2 \pm \sqrt{2V} \frac{dx}{dt} \right)$$

$$= \underbrace{\int_0^T dt \left(\frac{1}{2} (\dot{x} \mp \sqrt{2V})^2 \right)}_{\geq 0} \oplus \underbrace{\int_{x(0)=-a}^{x(T)=a} \sqrt{2V(x)} dx}_{\text{this is solely determined by B.C.}}$$

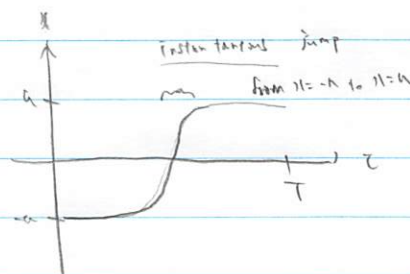
$-a \rightarrow a$ is $+a$ for $\dot{x} > 0$

↑ This is solely determined by B.C.

To minimize $SE(x)$ w/ $\begin{cases} x(0) = -a \\ x(T) = +a \end{cases}$,

we need to solve $\dot{x} = \pm \sqrt{2V(x)}$

$$\geq \int_{-a}^a \sqrt{2V(x)} dx = S_2 \quad (\text{instanton action})$$



$$(\langle a| \quad \langle a|) e^{-\hat{H}T} \begin{pmatrix} |a\rangle \\ |a\rangle \end{pmatrix} \\ = \begin{pmatrix} 1 & \mp e^{-\frac{S_1}{\hbar}} \\ \mp e^{-\frac{S_2}{\hbar}} & 1 \end{pmatrix}$$

The eigenvectors of this 2×2 matrix: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

eigenvalue: $1 \mp \mp e^{-\frac{S_1}{\hbar}}, 1 - \mp \mp e^{-\frac{S_2}{\hbar}}$
 A.S. 2^{nd} excited

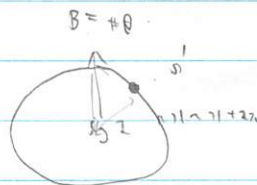
$$e^{-\hat{H}T} |\psi_{ns.}^{1^{st} \text{ excited}}\rangle = e^{-E_{ns.}^{1^{st}} T} \begin{pmatrix} E_{ns.}^{1^{st}} = 0 \mp \mp e^{-\frac{S_1}{\hbar}} \\ \ll 1/T \end{pmatrix} \\ \approx 1 \pm T \mp \mp e^{-\frac{S_1}{\hbar}} + O(T^2)$$

• DIGA for QM. for a particle on S^1 ,

$$H = \frac{1}{2} P_x^2 + V(x)$$

$$V(x+2\pi) = V(x)$$

$$(e.g. V(x) = -g \cos(Nx))$$



For QM on S^1 :

$\psi(x)$: wave function

$|\psi(x)|^2$: probability density for particle on x .

$$\psi(x+2\pi) = e^{i\theta} \psi(x)$$

One may multiply a U(1) phase factor

without affecting $|\psi(x)|^2$
 physical quantity.

One may eliminate this U(1) phase by redefining

$$\tilde{\psi}(x) = e^{i\theta \frac{x}{2\pi}} \psi(x)$$

$$x \rightarrow x+2\pi \Rightarrow \tilde{\psi}(x+2\pi) = \tilde{\psi}(x)$$

$$B = \delta A$$



$$\Rightarrow \hat{H}_0 = \frac{1}{2} \left(P_x + \frac{\hbar}{2\pi} \right)^2 + V(x)$$

$$\left(\begin{array}{l} \hat{H} \tilde{\psi} = E \tilde{\psi} \\ \tilde{\psi}(x+2\pi) = e^{i\theta} \tilde{\psi}(x) \end{array} \Leftrightarrow \begin{array}{l} \hat{H}_0 \tilde{\psi} = E \tilde{\psi} \\ \tilde{\psi}(x+2\pi) = \tilde{\psi}(x) \end{array} \right)$$

$$\langle x_f | e^{-\hat{H}_0} | x_i \rangle$$

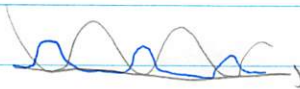
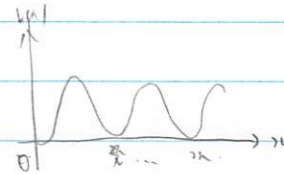
$$= \int_{x(0)=x_i}^{x(T)=x_f} \mathcal{D}x \, e^{-\int_0^T dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right) + i \frac{\theta}{2\pi} \int_0^T \dot{x} dt}$$

$$\left(\begin{array}{l} \langle p|x \rangle = e^{ipx} \\ \int dp \int dx \, e^{i \int_0^T \dot{x} - \int_0^T (H_0(x))} \end{array} \right) \quad \hbar \text{ units.}$$

$$V(x) = g(1 - \cos(Nx))$$

N "classical mags."

$$N = \frac{2\pi}{\lambda} \pm (2n, 2n+1, N-1)$$



⊗ This positive supersed state

is actually the ground S.S. for $-\pi < \theta < \pi$

Let's evaluate the $N \times N$ matrix elements

$$\langle x = \frac{2\pi}{N} k_2 | e^{-\hat{H}T} | x = \frac{2\pi}{N} k_1 \rangle$$

$k_1 = k_2$: we have a time-independent classical sol. $x(t) = \frac{2\pi}{N} k_1$

$$\Rightarrow \langle \frac{2\pi}{N} k_1 | e^{-\hat{H}T} | \frac{2\pi}{N} k_1 \rangle = 1$$

$$\underline{k_2 = k_1 \pm 1}: \quad \langle \frac{2\pi}{N} (k_1 \pm 1) | e^{-\hat{H}T} | \frac{2\pi}{N} k_1 \rangle$$

$$= \int_{x(0)=\frac{2\pi}{N} k_1}^{x(T)=\frac{2\pi}{N} (k_1 \pm 1)} \mathcal{D}x \, e^{-\int_0^T dt \left(\frac{1}{2} \dot{x}^2 + V(x) \right) + i \frac{\theta}{2\pi} \int_0^T \dot{x} dt}$$

$$S[x] = \int_0^T dt \left[\frac{1}{2} \dot{x}^2 + V(x) \right]$$

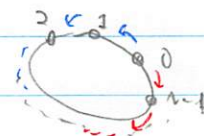
$$= \int_0^T dt \frac{1}{2} (\dot{x} \mp \sqrt{2V})^2 \pm \int_0^T dt \sqrt{2V} \dot{x}$$

$$\geq \int_{x=\frac{2\pi}{N} k_1}^{x=\frac{2\pi}{N} (k_1 \pm 1)} \sqrt{2V(x)} = S_{\pm}$$

$$+ i \frac{\theta}{2\pi} \int_0^T \dot{x} dt = i \pm \frac{2\pi}{N} \times \frac{\theta}{2\pi} = i \pm \frac{\theta}{N}$$

$$\left(\langle \frac{2\pi}{N} k_2 | e^{-\hat{H}T} | \frac{2\pi}{N} k_1 \rangle \right)_{k_1, k_2 = 0, 1, \dots, N-1}$$

$$= \begin{pmatrix} 1 & T \times \# e^{-\frac{\theta}{N}} e^{i\frac{\theta}{N}} & & \\ T \times \# e^{i\frac{\theta}{N}} e^{-\frac{\theta}{N}} & 1 & & \\ & & \ddots & \\ & & & T \times \# e^{i\frac{\theta}{N}} e^{-\frac{\theta}{N}} & 1 \end{pmatrix}$$



(with 1-inst. calculation.)

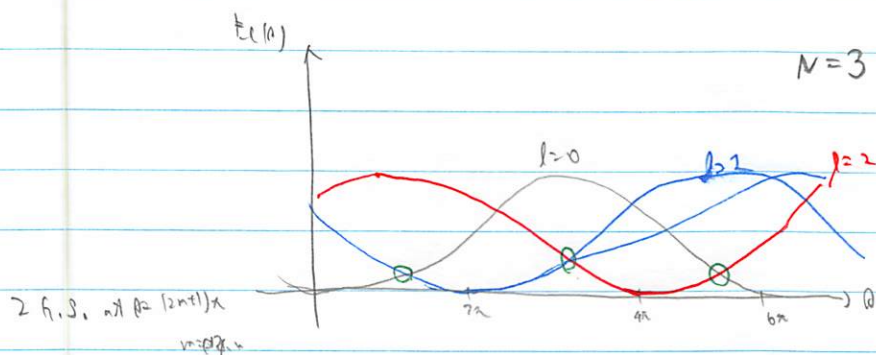
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N eigenvectors of this $N \times N$ matrix
($\omega = e^{\frac{2\pi i}{N}}$)

$$|\psi_l\rangle = \begin{pmatrix} 1 \\ \omega^l \\ \vdots \\ \omega^{(N-1)l} \end{pmatrix} \quad (l = 0, 1, \dots, N-1)$$

Corresponding eigenvalue

$$1 + T + T^2 + \dots + T^{N-1} = e^{\frac{2\pi i}{N}} \cos\left(\frac{\theta + 2\pi l}{N}\right) = -F_2(\theta)$$



$$T_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \psi_1 = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \Rightarrow \psi_2 = \begin{pmatrix} 1 \\ \omega^2 \end{pmatrix} \Rightarrow \psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

ψ_0 jumps at $\theta = \pi$ ψ_1 jumps at $\theta = 2\pi$ ψ_2 jumps at $\theta = 4\pi$

Why do we have level crossing in this model?

$$\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \rightarrow \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$$

ψ_0 jumps at $\theta = \pi$ ψ_1 jumps at $\theta = 2\pi$ ψ_2 jumps at $\theta = 4\pi$

Answer: Because they are protected by the symmetry

$$S(x) = \int \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} (1 - \cos \mu x) \right) dx + i \frac{\theta}{2\pi} \int dx$$

$$\psi_{\text{sym}} = x \mapsto x + \frac{2\pi}{N}$$

(If we repeat this operation N times, $x \mapsto x + 2\pi$)

$$\psi_{\text{sym}} |x\rangle \rightarrow |x + \frac{2\pi}{N}\rangle$$

$$|\psi_l\rangle = \sum_{k=0}^{N-1} \omega^{kl} |x = \frac{2\pi}{N} k\rangle$$

$$\Rightarrow \sum_{k=0}^{N-1} \omega^{k(l-1)} |x = \frac{2\pi}{N} (k+1)\rangle$$

$$= \omega^{-l} |\psi_l\rangle$$

so the level crossing is not a real crossing, it's just a level repulsion.

dilute gas app.

From now, we give a rederivation of this result from the computation of the partition function:

$$\mathcal{Z}(\beta) = \text{Tr}_{\mathcal{H}} (e^{-\hat{H}T})$$

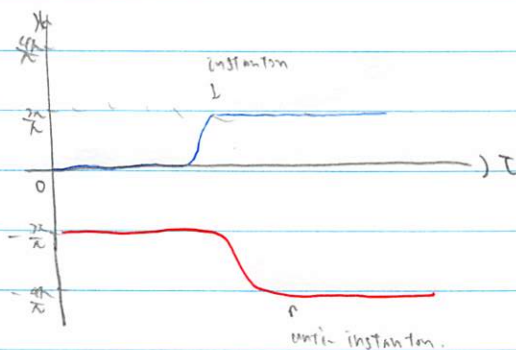
From our computation just done, we know this answer.

$$= \sum_{l=0}^{\infty} \exp \left(-T \times \left(-2 \mp e^{\frac{\beta}{T}} \cos \left(\frac{\beta + 2\pi l}{N} \right) \right) \right)$$

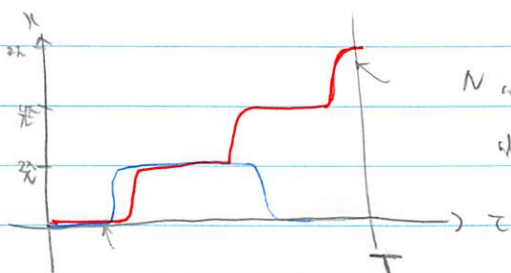
$$= \int_{\substack{\chi(\tau) = \chi(0) \\ \text{periodic b.c.}}} \mathcal{D}\chi \exp \left[- \int_0^T d\tau \left(\frac{1}{2} \dot{\chi}^2 + V(\chi) \right) + i \frac{\theta}{2\pi} \int d\tau \right]$$

Instanton

1 instanton event. 境界条件 端点条件.



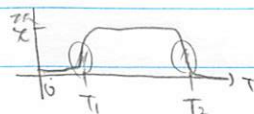
- What kind of configurations are consistent w/ Periodic B.C.?

 $N=3$ N consecutive events of instanton.also satisfies P.B.C. (mod 2π)

Instanton - anti-instanton event satisfies P.B.C.

Let's evaluate how these events contribute to $\mathcal{Z}$.

Instanton - anti-instanton

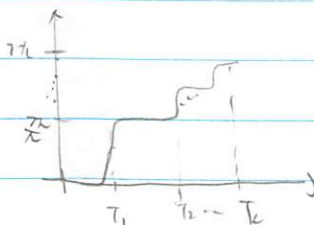


Boltzmann weight

$$S[\chi] = \left(\frac{S_1}{T} + i \frac{\theta}{N} \right) + \left(\frac{S_2}{T} - i \frac{\theta}{N} \right) = 2 \frac{S_2}{T}$$

 $\Downarrow$

$$\mp e^{\pm \frac{\theta}{N}} \int_{\chi(\tau_1)=\chi(\tau_2)} \mathcal{D}\chi = \frac{T^2}{2} \mp e^{\pm \frac{\theta}{N}}$$

N consecutive instanton events

$$S[x] = \left(\frac{S_2}{\hbar} + i\frac{\theta}{\hbar} \right) N = N \frac{S_2}{\hbar} + i\theta$$

↓
Boltzmann weight

$$\# e^{-N \frac{S_2}{\hbar} + i\theta} \times \int_{0 \leq \tau_1 < \dots < \tau_N \leq T} d\tau_1 \dots d\tau_N$$

(classical)

Grand-canonical ensemble of
 N -instanton.

↓

$$\frac{T^N}{N!} \# e^{-N \frac{S_2}{\hbar} + i\theta}$$

$$\mathcal{Z}(\theta) = \int dx \exp \left(- \int dx \left(\frac{1}{2} \dot{x}^2 + g(r - r_0(x)) + \frac{i\theta}{2\pi} \int dx \right) \right)$$

$\cong$ classical stat. mech. for instanton and anti-instanton.

$$= \sum_{\substack{n, \bar{n} \\ \# \text{ of instanton} \\ \# \text{ of anti-instanton}}} \frac{1}{n! \bar{n}!} \left(T \# e^{-\frac{S_2}{\hbar} + i\frac{\theta}{\hbar}} \right)^n \left(T \# e^{-\frac{S_2}{\hbar} - i\frac{\theta}{\hbar}} \right)^{\bar{n}} \times \int_{n, \bar{n}} d\mu$$

$$= \sum_{n, \bar{n} \geq 0} \frac{1}{n! \bar{n}!} \left(T \# e^{-\frac{S_2}{\hbar} + i\frac{\theta}{\hbar}} \right)^n \left(T \# e^{-\frac{S_2}{\hbar} - i\frac{\theta}{\hbar}} \right)^{\bar{n}} \\ \times \sum_{l=0}^{n-1} e^{\frac{2\pi i}{\hbar} (n-\bar{n})l}$$

$$= \sum_{l=0}^{n-1} \left(\sum_{n, \bar{n} \geq 0} \frac{1}{n! \bar{n}!} \left(T \# e^{-\frac{S_2}{\hbar} + i\frac{\theta + 2\pi l}{\hbar}} \right)^n \left(T \# e^{-\frac{S_2}{\hbar} - i\frac{\theta + 2\pi l}{\hbar}} \right)^{\bar{n}} \right) \\ = \sum_{l=0}^{n-1} \exp \left(- T \# e^{-\frac{S_2}{\hbar}} \left(e^{i\frac{\theta + 2\pi l}{\hbar}} + e^{-i\frac{\theta + 2\pi l}{\hbar}} \right) \right) \\ \underbrace{\qquad\qquad\qquad}_{2 \cos \left(\frac{\theta + 2\pi l}{\hbar} \right)}$$

$\Rightarrow$ We reproduced the result!!

$$\mathcal{Z}(\theta) = \mathcal{Z}(0) \rightsquigarrow$$

As a quantum system $(H = [2] \text{ gl})$ (1-10)

the system at $\theta = 2\pi$ and θ

$$\text{the unitary equivalent } H_{\theta} = U^\dagger H_0 U$$

↑
unitary eq. for H .

2.2d U(1) gauge theories

on $P^2 \times \mathbb{R}$ $\Rightarrow$ $\left\{ \begin{array}{l} \text{4d free Maxwell (pure U(1) gauge theory)} \\ \text{4d U(1) gauge + charge -N Higgs} \\ \cong \text{particle on } S^1 \text{ w/ } v(x) \sim g(1 - \cos(Nx)) \end{array} \right.$

on $P^2 \times T^2$ $\Rightarrow$ $\left\{ \begin{array}{l} \text{4d U(1) gauge theory + Dirac fermion} \\ \text{= Schwinger model} \\ \text{4d Schwinger model + charge -N Higgs} \end{array} \right.$

• 2d free Maxwell theory.

Convention U(1) gauge field $A = A_\mu dx^\mu$

Gauge transformation $A_\mu \rightarrow A_\mu - i e^{-i\lambda} d_\mu e^{i\lambda}$

$\lambda \sim \lambda + 2\pi$



U(1) gauge transformation

$\left\{ \right.$

Matter ψ : covariant derivative

$$D_\mu \psi = (\partial_\mu + i A_\mu) \psi$$

$\uparrow$ we do not write coupling constant

Lagrangian (Minkowski) (QED)

$$\mathcal{L} = \frac{1}{2} F_\mu F^\mu + \bar{\psi} \gamma^\mu (\partial_\mu + i A_\mu) \psi$$

gauge coupling const. $\frac{1}{2} F_\mu F^\mu = \frac{1}{2} F_\mu F_\mu$

In this convention, we have a canonical normalization of the Dirac quantization.



Let's consider a U(1) gauge field on S^2 .

northern hemisphere.

For gauge field on a manifold w/ non-trivial topology.



the gauge field is NOT a globally defined vector field

southern hemisphere.

On each local coordinate, (i.e. northern or southern hemisphere)

we have a vector potential $\left\{ \begin{array}{l} A_\mu^N \\ A_\mu^S \end{array} \right\}$

On equator: $A_\mu^N = A_\mu^S - i e^{-i\lambda} d_\mu e^{i\lambda}$

A_μ^N, A_μ^S are related by some gauge transformation.

9.

$$\begin{aligned}
 \int_{S^2} \vec{F} \cdot d\vec{A} &= \int_{\text{top hemisphere}} dA^N + \int_{\text{bottom hemisphere}} dA^S \\
 &= \int_{\text{top hemisphere}} A^N - \int_{\text{bottom hemisphere}} A^S \\
 &= \int_{S^2} (A^N - A^S) \\
 &= \int_{S^2} d\lambda \cdot 2\pi \quad \leftarrow \text{magnetic charge inside } S^2 \\
 &\quad q \sim \lambda / 2\pi
 \end{aligned}$$

We consider following Lagrangian as a 2d Maxwell:

$$\begin{aligned}
 \mathcal{L}(\text{Maxwell}) &= \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} \\
 \text{d Euclid} &= \frac{1}{2g^2} \dot{A}_1^2 + \frac{1}{2\pi} \dot{A}_1 \quad (\text{in } A_0 = 0 \text{ gauge})
 \end{aligned}$$

$$\mathcal{L}_E = \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i \frac{1}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}$$

We can easily solve this model as $\mathcal{L}(\text{particle})$ is quadratic in A_1 :

$$\pi = \frac{\delta \mathcal{L}_{\text{Maxwell}}}{\delta \dot{A}_1} = \frac{1}{g^2} \dot{A}_1 + \frac{1}{2\pi}$$

$$E = \pi \dot{A}_1 - \mathcal{L}_{\text{Maxwell}} = \frac{g^2}{2} \left(\pi - \frac{1}{2\pi} \right)^2$$

$$\frac{\delta \mathcal{L}_{\text{Maxwell}}}{\delta A_0} \Big|_0 = 0$$

$$\frac{\delta}{\delta \pi} \pi = 0$$

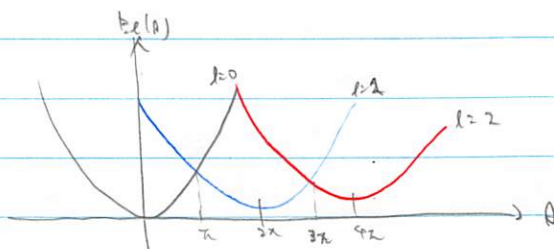
(Gauss' law constraint)

$$\chi(A_1 + d\lambda) = \chi(A_1)$$

We now consider the compactified space $\text{Phase} \times S^1$

$$\chi_e(A) = e^{i\lambda} S_1 A$$

$$\sim \chi_e(A) = E_e(A) \chi_e(A) \quad \therefore E_e(A) = \frac{g^2}{2} \left(1 - \frac{1}{2\pi} \right)^2$$



Interpretation $E = \dot{A}_1^2 = g^2 \left(\pi - \frac{1}{2\pi} \right)^2$

$$\left(\pi - \frac{1}{2\pi} \right) = \text{background electric field}$$

$\oplus \xrightarrow{\quad} \pi \oplus$ charge on infinities.
 $-2 \xrightarrow{\quad} +2$
 $E_{\text{from}} = g^2 \left(1 - \frac{1}{2\pi} \right)^2$

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We can obtain this result from each integral.

$$Z(0) = \int \mathcal{D}A \exp \left(-\frac{1}{2g^2} \int dA \wedge \star dA + i \frac{\theta}{2\pi} \int dA \right)$$

$$= \left(\frac{1}{\sqrt{\det(-\Delta)}} \right) \sum_{m \in \mathbb{Z}} \exp \left(-\frac{1}{2g^2} \left(\frac{2\pi m}{\text{Vol}} \right)^2 \times \text{Vol} + i \frac{\theta}{2\pi} 2\pi m \right)$$

Poisson summation
formula

$$\sum_{m \in \mathbb{Z}} \exp \left(-\text{Vol} \left(\frac{1}{2} + \frac{\theta^2}{2} \left(i - \frac{\theta}{2\pi} \right)^2 \right) \right)$$

$E_{\text{eff}}(\theta)$

• 2d charge -N Abelian-Higgs model

$$\mathcal{L} = \frac{1}{2g^2} \int dA \wedge \star dA - i \frac{\theta}{2\pi} \int dA \quad \leftarrow \text{Maxwell}$$

$$+ \int d^2x \left\{ |d_\mu + N A_\mu \Phi|^2 + \frac{\lambda}{4} (|\Phi|^2 - v^2)^2 \right\}$$

• ($v^2 < 0$) $\rightsquigarrow$  $\bar{\Phi}$ is massive

$\rightsquigarrow$ Integration over Φ unifies g^2

$\Rightarrow$ we are back to the pure Maxwell.

(Φ is not renormalized because Φ is C-odd)

• ($v^2 > 0$) - Higgs phase

The potential is minimized when $|\Phi| = v$

$\rightsquigarrow$ we parametrize this classical vac. by

$$\Phi = v e^{i\phi(x)}$$

By substituting this expression,

$$|d_\mu + iN A_\mu \Phi|^2 \times \text{Vol} = v^2 |d_\mu \phi + N A_\mu|^2 \quad \text{mass for gauge field via Anderson-Higgs mechanism.}$$

This suggests that A_μ at the classical vac. is given by

$$A_\mu = -\frac{1}{N} \partial_\mu \phi \quad (A = -\frac{1}{N} d\phi)$$

For this config, $F = \star dA = 0 \rightsquigarrow$ field strength = 0 (mag. field)

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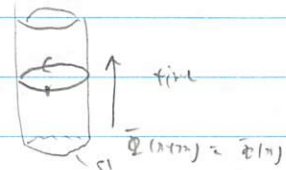
Let's consider the case when our 2d spacetime is $\mathbb{R} \times S^1$
(Euclidean) time space.

For the above classical vacuum:

$$e^{i \int_{S^1} A} = e^{i \int_{S^1} (-\frac{1}{\kappa} d\phi)} = e^{-\frac{i}{\kappa} \int_{S^1} d\phi} = e^{-\frac{2\pi i}{\kappa} k}$$

Spatial Wilson loop
= Aharonov-Bohm phase

$$k = 0, 1, \dots, N-1$$



$$\begin{aligned} \bar{\phi}(x+n) &\sim \bar{\phi}(x) \\ \Rightarrow e^{i\bar{\phi}(x+n)} &= e^{i\bar{\phi}(x)} \\ &= \bar{\phi}(x) = \phi(x) \pmod{2\pi} \end{aligned}$$

• We now obtain

N classical vacua

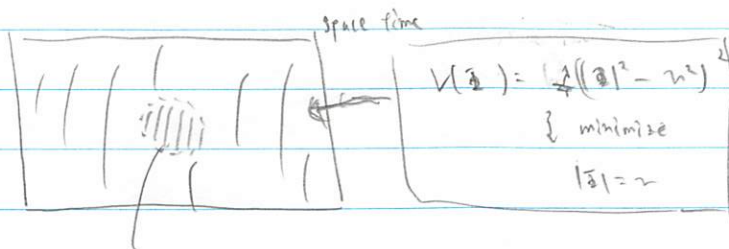
$$|k\rangle \rightarrow e^{i \int_{S^1} A} = e^{-i \int_{S^1} \frac{d\phi}{\kappa}} = e^{-i \frac{2\pi k}{N}}$$

• In analogy w/ $\theta\mu$ on S^1 ,

there exists an instanton process for $e^{-\tilde{H}T}$

describing $|k\rangle \rightarrow |k\pm 1\rangle$ ② What is this object here?

Vortex in Abelian-Higgs model



We allow excitation for S_z
in a bounded region

In the outside region, we have the classical vacuum,

$$\bar{\phi}(x) = v e^{i q(x)}, \quad A = -\frac{1}{\kappa} d\phi.$$

What is the property of these objects?

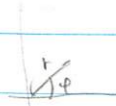
magnetic flux is quantized in the unit of $\pm \frac{1}{N}$

$$\begin{aligned} \frac{1}{2\pi} \oint_{\partial D} A &= \frac{1}{2\pi} \int_D dA = \frac{1}{2\pi} \int_D d\left(-\frac{1}{\kappa} d\phi\right) \\ &= \frac{1}{2\pi} \int_D d\left(-\frac{1}{\kappa} d\phi\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi} \int_{S^1} A = -\frac{1}{2\pi N} \int_{S^1} d\phi \in \frac{1}{N} \mathbb{Z} \\ &= \left(-\frac{m}{N}\right) \end{aligned}$$

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For $m=1$ $\phi(r) = m \varphi$
 $= \varphi$



In blue region $\bar{\phi}(r, \varphi) = n e^{i\varphi}$

If we naively extend this expression

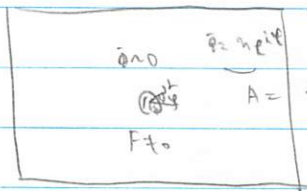
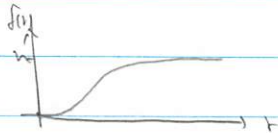
to the vicinity of the origin

We have a phase inconsistency for $\bar{\phi}$ at $r=0$

(φ is ill-defined at $r=0$ so $\bar{\phi} = n e^{i\varphi}$ should not have a dep. at $r=0$)

$\Rightarrow \bar{\phi}(r, \varphi) = f(r) e^{i\varphi}$

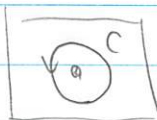
$f(r \rightarrow \infty) = n$ classical vac for space time ∞
 $f(r \rightarrow 0) = 0$ for consistency to evade phase ambiguity.



or vortex object that carries
magnetic flux $\pm \frac{1}{N}$

This actually gives an instanton tunneling term: $|1\rangle \Rightarrow |k+1\rangle$

To see this, let's surround this vortex w/ Wilson loop



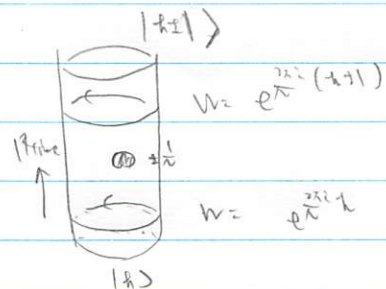
$w(C) = e^{i \oint_C A}$
 $= e^{i \frac{1}{2\pi} \oint_C d\varphi} = e^{i \frac{2\pi}{2\pi}}$

$\Rightarrow$ If the vortex lies inside the Wilson loop,
it rotates its phase by $e^{i \frac{2\pi}{N}}$

By going back to the $1/r S^1$ geometry

we find that the Wilson loop along S^1

is related by $e^{\pm \frac{2\pi i}{N}}$ via the vortex



$\Rightarrow$ On $1/r S^1$, we can evaluate the transition amplitude
in the same manner for QM.

using the dilute water (instantaneous) gas approximation

$$M = T^2 = S' \times S' \quad \text{with volume } V_{\text{el}}$$

$$Z = \int dA \, d\bar{A} \, e^{-S(A, \bar{A})} = \int dA \, \rho(A)$$

$\cong$ classical 2d stat. mech. for vortex gas.

$$= \sum_n \frac{1}{n! \bar{n}!} \left(\text{vol} \# e^{-S_n + i \frac{Q}{N}} \right)^n \left(\text{vol} \# e^{-S_{\bar{n}} - i \frac{Q}{N}} \right)^{\bar{n}}$$

$\theta(\vec{r}) \geq 0$
 # of vortex $\rightarrow$ # of anti-vortex
 $\frac{1}{2\pi} \oint \vec{A} \cdot d\vec{l} = \frac{1}{N}$

$\oint \vec{A} \cdot d\vec{l} \in \mathbb{N} \frac{2\pi}{N}$
 $\sum_{l=0}^{N-1} e^{\frac{2\pi i}{N} (n - \vec{n}) l}$

2nd periodic $\theta, \vec{r}$
 $\frac{1}{2\pi} \oint \vec{A} \cdot d\vec{l} = \frac{1}{N}$

$$= \sum_{l=0}^{N-1} \exp \left[-\log r \left(-z \neq e^{i\theta_l} \cos \left(\frac{\theta_l + 2\pi l}{N} \right) \right) \right]$$

↑
We obtain the same expression with the case of ∂M on S^1

Lattice theory

$$S = \sum_p \left(\bar{\psi}_p \gamma^\mu \psi_p \right) + \sum_n \left(\bar{\psi}_n \gamma^\mu \psi_n \right) + \sum_n \left(\bar{\psi}_n \gamma^\mu \psi_n \right) + \sum_n \left(\bar{\psi}_n \gamma^\mu \psi_n \right)$$

$$\begin{cases} B_T \rightarrow B_T (h^2)_T \\ U_{T+1} \rightarrow p^{i \frac{7K}{20}} U_T \end{cases} \quad \text{= } U_T \text{ form symmetry.}$$

$$Z_{\text{out}}(s) = e^{i\frac{\pi}{2}} \text{SDP} \quad Z_0(\text{SDP})$$

N quantum vacua belong to

bestimmt eigenvalue der $2n$ 1-form sym.

• Semiclassics for 4d $SU(N)$ YM on $\mathbb{R}^2 \times (T^2)_{\text{torus}}$

$$\left(\begin{array}{l} \text{Ref: } 2201.06166 \\ 2405.12462 \end{array} \quad \left(\begin{array}{l} \text{2d: } \text{Schwinger + charge-N Higgs theory} \\ \text{YM \& } \text{CS} \text{ (5d)} \end{array} \right) \right)$$

• Brief review on 4d $SU(N)$ YM

$$S_E = \underbrace{\frac{1}{g^2} \int \text{tr} (f \wedge f)}_{\text{kinetic term } (\vec{E} \cdot \vec{B})} + i \underbrace{\frac{\theta}{2\pi} \int \text{tr} (F \wedge F)}_{\text{top term } \vec{E} \cdot \vec{B}}$$

fundamental field, $SU(N)$ gauge field

$$a = a_\mu dx^\mu$$

lie $(SU(N))$ -valued 1-form locally

field strength

$$f = da + i a \wedge a, \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + i [a_\mu, a_\nu]$$

topological sector for $SU(N)$ YM

Fact: On any closed 4d manifold M ,

$$\frac{1}{2\pi^2} \int_M \text{tr} (f \wedge f) \in \mathbb{Z} \quad \leftarrow \text{instanton number}$$

Especially

on $\mathbb{R}^4$:

To have finite action, $f \rightarrow 0$ at infinities

$\Leftrightarrow a \rightarrow$ pure gauge

$$= -iV^\dagger dV$$

for some $SU(N)$ -valued scalar $V = \int_{S^3_\infty} \text{tr} (V^\dagger dV) \in \pi_3(SU(N))$

$$\frac{1}{2\pi^2} \int_{\mathbb{R}^4} \text{tr} (f \wedge f) = \frac{1}{2\pi^2} \int_{S^3_\infty} \text{tr} \left(a da + \frac{2}{3} i a^3 \right)$$

$\leftarrow$ At infinities

$$a = -iV^\dagger dV$$

$$= \frac{1}{24\pi^2} \int_{S^3_\infty} \text{tr} (V^\dagger dV)^3$$

$\in \pi_3(SU(N))$ $\leftarrow$ How many times V wind $SU(N)$ along S^3_∞ .

$$\left(\begin{array}{l} \text{Remark: } \underbrace{SU(N)}_V \cong S^3 \\ \leadsto \pi_3(SU(N)) \text{ just counts the winding \# from } S^3 \text{ to } S^3. \end{array} \right)$$

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Bogomolnyi - trick :

→ we can find the first-order PDE

for minimizing $P(B)$ on each instanton sector

$$\begin{aligned} P_e(S_n) &= \frac{1}{g^2} \int \text{tr} (f \wedge * f) + \frac{1}{2} |E - B|^2 \geq 0 \\ &= \frac{1}{2g^2} \int \text{tr} (f \wedge * f) \wedge * (f \wedge * f) \\ &\quad \pm \frac{g^2}{2} \left[\frac{1}{8\pi^2} \int \text{tr} (f \wedge f) \right] \end{aligned}$$

← this part is an integer determined by P.C. at ∞ .

→ we obtain "self-dual" YM eq.

$$f = \pm * f$$

→ For 1-instanton $\frac{1}{8\pi^2} \int \text{tr} (f \wedge f) = 1$

Boltzmann weight is given by $e^{-\frac{8\pi^2}{g^2}}$

Remark In Hamilton formalism for YM w/ $A_0 = 0$ gauge

we have two equivalent way to introduce θ :

$$\begin{aligned} \textcircled{1} \quad \hat{H} &= \frac{g^2}{2} \hat{\pi}^2 + \frac{1}{2g^2} B^2 \\ H &= \{ \hat{H} (A) \} \end{aligned}$$

gapped gauge field Gauss law $\hat{H}(A^0) = \hat{H}(A)$ for small gauge transformation, g

For large gauge trans g :

$$\{ \hat{H}(A^0) \} = e^{i\theta \frac{1}{24\pi^2} \int \text{tr} (B \wedge B)^3} \cdot \hat{H}(A)$$

$$\textcircled{2} \quad \hat{H}_\theta = \frac{g^2}{2} \left(\hat{\pi} + \frac{e}{2\pi} B \right)^2 + \frac{1}{2g^2} B^2$$

↑
Witten effect

$\hat{H}_\theta = \{ \hat{H}_\theta(A^0) \} = \hat{H}$ is gauge inv for both small and large gauge transformations.

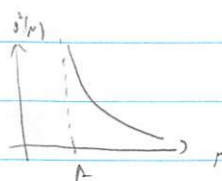
• 1-loop RG for YM

$g^2(\mu) \sim \mu$: renormalization scale.

string scale via dimensional transmutation

$$e^{-\frac{8\pi^2}{g^2(\mu)}} = \left(\frac{\Lambda}{\mu} \right)^{\frac{11}{3}N} \left(\frac{4\pi}{3} \right)^{\frac{N}{2}} \quad \text{for } \mu \gg \Lambda$$

fluctuation determinant



→ At low energies, g and YM becomes strongly coupled.

It's a tough problem to identify its vacuum structure.

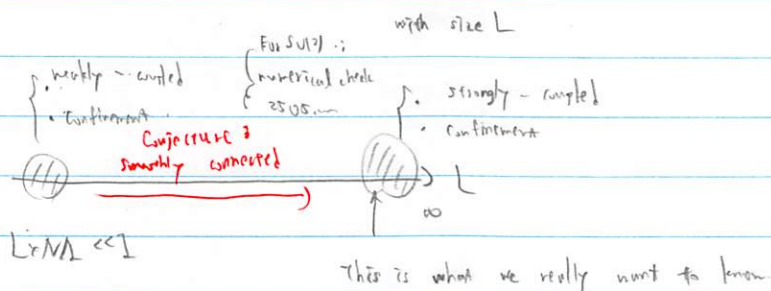
To solve 4d YM analytically,

We need to discover a clever setup s.t.

- we deform the setup without affecting the confinement vacuum structure,
- the deformed theory is weakly coupled $\Rightarrow$ analytically solvable.

In this lecture, we especially focus on

YM on $\mathbb{R}^2 \times (T^2)$ sym twisted B.C.



• YM on $\mathbb{R}^3 \times S^1$ (thermal YM)

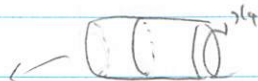
$$A_\mu(x, y_4 + \beta) = A_\mu(x, y_4)$$

$$A_4 = \frac{1}{\beta} \phi(x) \quad \text{diagonal} \quad \text{is } U(1) \text{ gauge}$$

$$= \frac{1}{\beta} \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_N \end{pmatrix}$$

$$P = P e^{i \oint A_4 dx_4}$$

$$= \begin{pmatrix} e^{i\phi_1} & & \\ & e^{i\phi_2} & \\ & & \ddots \\ & & & e^{i\phi_N} \end{pmatrix}$$



$\text{tr}(P)$ is order parameter for $U(1)$ (center) conf. sym.

$$\sim \begin{cases} \langle \text{tr}(P) \rangle = 0 & \Leftrightarrow \text{confinement} \\ \langle \text{tr}(P) \rangle \neq 0 & \Leftrightarrow \text{deconfinement} \end{cases}$$

At the classical level, any values of ϕ have same energy.

Fact: within 1-loop effective action (Gross-Neveu-Yukawa)

$$V_{1\text{-loop}}(\phi) \propto -\frac{1}{24} \sum_{n=1}^{\infty} \frac{1}{n^4} |\text{tr}(P^n)|^2$$

$$\Rightarrow P = e^{i \frac{2\pi}{N} \sum_{a=1}^{N-1} a} \quad (a=0, \dots, N-1) \quad \text{are the minimum.}$$

center-hidden vacua.

Physical interpretation:

With a given $\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$ background,

$$|\text{tr}(P^n)|^2 = \left| \sum_{i,j} \frac{1}{n} (\phi_i - \phi_j) a_n^{i,j} \right|^2 \quad \begin{aligned} & \text{off-diagonal gluons } a_n^{i,j} \text{ acquire} \\ & \text{the mass } \propto \frac{|\phi_i - \phi_j|}{n} \\ & \text{Is } \phi_i = \phi_j \text{ for all } i, j, \quad \text{all massless adj gluons are} \\ & \Rightarrow \text{Maximized} \quad \text{1-loop energy maximized.} \end{aligned}$$

T/μ
 T_c

$(\chi_n^{(1)})_{3d} \times (\chi_n^{(1)})_{2d}$
 Spatial Wilson loop W confined Polyakov loop P deconfined.

$(\chi_n^{(1)})_{4d}$ is totally unbroken
 $\langle W(1) \rangle \sim \text{area law}$

- Even though we can introduce a high-energy scale by considering $R^3 \times (S^1)_p$ with small p , such a region is separated from 4d gauge vacuum via confinement-deconfinement transition.

- 4d $SU(N)$ YM on $R^2 \times T^2$ w/ 't Hooft twisted B.C.
 $(x_3, y_4) \sim (x_3+L, y_4+L)$

2 ways to understand 't Hooft twisted B.C.

- ① $SU(N)$ clock & shift matrix:

$$C = \begin{pmatrix} 1 & \omega & 0 \\ 0 & \ddots & \omega^{N-1} \\ 0 & & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & & & 0 \end{pmatrix}$$

$\uparrow$ twist for x_4 -direction $\uparrow$ twist for x_3 -direction

$$\begin{cases} A_\mu(x_1, x_2+L, x_4) = S^{-1} A_\mu(x_1, x_2, x_4) S \\ A_\mu(x_1, x_3, x_4+L) = C^{-1} A_\mu(x_1, x_3, x_4) C \end{cases}$$

This twisted B.C. cannot be transformed to periodic B.C. via gauge transformation, because $CS = \omega S C$ ($\omega = e^{2\pi i/N}$).

- ② Wilson lattice gauge theory

$$S_{\text{lattice}}[U_\mu] = -\beta \sum_{\text{plaq}} \text{tr}(U_{\text{plaq}}) + \text{c.c.}$$

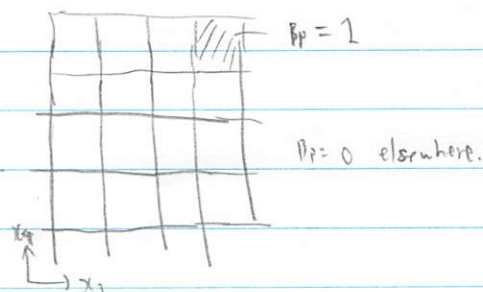
$$U_{\text{plaq}} = P \prod_{\text{plaq}} U_\mu \leftarrow \text{path-ordered product along plaquettes.}$$

periodic B.C.

$\Downarrow$ 't Hooft twisted B.C.
(= 't Hooft flux)

$$S[U_\mu, B_p] = \sum_{\text{plaq}} \left(e^{2\pi i B_p} \text{tr}(U_{\text{plaq}}) + \text{c.c.} \right)$$

$y, t, x_4 \sim y+L, t+L, x_4+L$



We set $B_p = 0$ except $\nabla \mu$

For $\nabla \mu$, $B_p = 1$

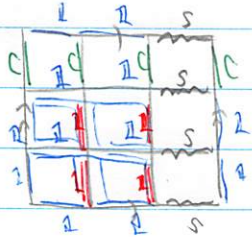
In order to see the appearance of CS mass, let's minimize the lattice action $S_{\text{lattice}}(U_\mu, B_p)$

sublattice condition
 $\left(= e^{2\pi i B_p} \text{tr}(U_{\text{plaq}}) = N \text{ for all plaquette.} \right)$

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$$\Leftrightarrow \begin{cases} \text{For } i \neq j, & \text{tr}(U_i) = N \text{ i.e. } U_i = \mathbb{1} \\ \text{For } i = j, & e^{\frac{2\pi i}{N}} \text{tr}(U_i) = N \text{ i.e. } U_i = e^{\frac{2\pi i}{N}} \mathbb{1}_p \end{cases}$$

We take the complete
axial gauge
along the blue line.



$$S C S^{-1} C^{-1} = e^{-\frac{2\pi i}{N}}$$

$$\Leftrightarrow S C = e^{-\frac{2\pi i}{N}} C S$$

$$(S, C \in \text{SU}(N))$$

U_p is conjugate transformation $(S, C) \mapsto (U S U^{-1}, U C U^{-1})$

$$S = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ & & & \\ & & 0 & \end{pmatrix}; \quad C = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \omega \end{pmatrix}$$

ed SU(N) = $\mathbb{Z}_N$ sym $\sim$ $\mathbb{Z}_N$ 2-form background gauge field

$$\left(\begin{array}{l} \text{On } \mathbb{R}^3 \times S^1 \times S^1 \\ P_{\text{cl}} = P_{\text{cl}} + A_3 \frac{dx^3}{L} + A_4 \frac{dx^4}{L} + \left(\frac{2\pi m_{\text{cl}}}{L^2} \right) dx^3 dx^4 \sim \int_{T^2} B = m_{\text{cl}} \\ \text{Higgs flux} \end{array} \right)$$

On this classical vacuum:

$$\begin{cases} P_3 = P e^{i \int_0^L dx^3 A_3} = S \\ P_4 = P e^{i \int_0^L dx^4 A_4} = C \end{cases}$$

We now find that $\mathbb{Z}_N \times \mathbb{Z}_N$ v-form center sym. is unbroken:

$$\text{tr}(P_3) = 0, \text{tr}(P_4) = 0$$

$$\left(\text{More generally, } \text{tr}(P_3^{k_3} P_4^{k_4}) = 0 \text{ for } (k_3, k_4) \neq (0, 0) \text{ mod } N \right)$$

$$\text{tr}(P_3) = \text{tr}(P_3 P_4 P_4^{-1})$$

$$= \text{tr}(P_4^{-1} P_3 P_4) = e^{\frac{2\pi i}{N}} \text{tr}(P_3) \rightarrow \text{tr}(P_3) = 0$$

Remark

On $\mathbb{R}^3 \times S^1$: classical $\rightarrow$ P_{cl} is arbitrary

1-loop analysis: P_{cl} selects center-broken vacua

On $\mathbb{R}^3 \times T^2$: classical $P_3 = S, P_4 = C$ $\rightarrow$ center symmetric vacuum is preferred

$$\left(\begin{array}{l} \text{In large } N, \text{ there is a subtlety of this classical stability} \\ \rightarrow \text{Hayashi, Ueno, 2005} \end{array} \right)$$

Now, we start considering the dynamics for T^2 direction

$$\alpha = \begin{cases} a_{2d} \sim 2d \text{ vector} \\ a_3, a_4 \sim 2d \text{ scalar} \end{cases} \quad (P_3, P_4) \sim (S, C)$$

$$|F_{\mu\nu}|^2 \supset |F_{23}|^2 + |F_{34}|^2 \quad , 2d \text{ gluon mass}$$

$$m_{2d}^2 \simeq \left(\frac{2\pi}{\Lambda L} n_3\right)^2 + \left(\frac{2\pi}{\Lambda L} n_4\right)^2 \quad (n_3, n_4) \neq (0, 0)$$

$\Rightarrow$ we do not have 2d zero mode at all.

$$C = \begin{pmatrix} e^{i\phi_1} & \\ & e^{i\phi_2} \end{pmatrix} = \begin{pmatrix} 1 & w_1 \\ & w_2 \end{pmatrix}$$

$$\phi_1 - \phi_2 = \frac{2\pi}{\Lambda} (i - \bar{i})$$

$$\text{if } m_{2d} = \frac{2\pi}{\Lambda L} (i - \bar{i})$$

n_3
 n_4

$\Rightarrow$ 2d - diagonal gluon gets a KK mass. i.e. $P_4 = C$

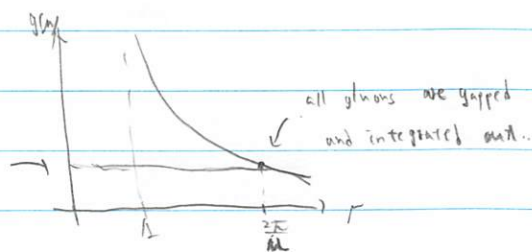
$$a_\mu(x_1, x_2, y_4 + L) = C^{-1} a_\mu(x_1, x_2, y_4) C$$

$$\text{Also for } x_3 \text{ direction, } a_\mu(x_1, y_3 + L, y_4) = S^{-1} a_\mu(x_1, y_3, y_4) S$$

$\Rightarrow$ Also for diagonal component

the KK gap $\frac{2\pi}{\Lambda L}$ is induced.

Let's now revisit the 1-loop running of $g(\mu)$



We obtain

finite and weak

2d coupling

$$\text{as long as } \Lambda \ll \frac{2\pi}{\Lambda L}$$

$\Rightarrow$ weakly-coupled 2d effective theory.

$$\bullet \text{ We confirmed } \text{tr}(P_3) = \text{tr}(P_4) = 0$$

i.e. confinement for T^2 - direction.

What about confinement for the T^2 direction?

i.e. Does $\langle w(c) \rangle$ obey area law?

inside T^2

$\bullet$ At the classical level, $\langle w(c) \rangle$ is perimeter law.

$\Rightarrow$ By taking into account the semiclassical contribution, this perimeter law becomes the area law.

- ② Key player for the semiclassical analysis.
 • Center - vortex fractional instanton

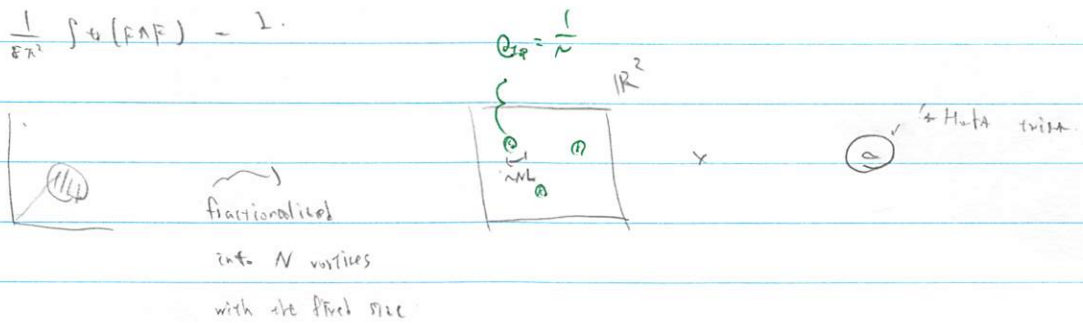
To do

Solve the self-dual YM eq. $F = *F$ for $R^2 \times T^2$ twist
 $\Leftarrow$ No one has closed analytic solution.

- ③ Numerical fact (Anzures - Araya, Monte '99)

4d instanton

$$\frac{1}{8\pi^2} \int d^4x (F \wedge F) = 1.$$



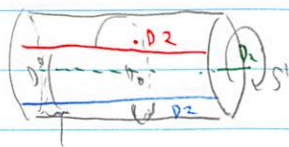
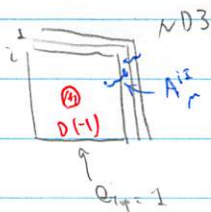
$\Leftarrow$ This is a consequence of the perturbative gap $\sim \frac{2\pi}{N}$

- ④ "Semi"-analytic understanding of this fractional instanton.
 (2405, ... ~ Y. Hara)

$$R^4 \Rightarrow R^3 \times \underbrace{S^1_{L_4}}_{T\text{-dual}} \Rightarrow R^2 \times (S^1_{L_3} + S^1_{L_4}) \text{ twist} \quad L_3 \gg NL_4 \quad \kappa \cdot L_3 = \frac{2\pi}{N}$$

$$F = *F \quad \text{w/ } Q_{top} = 1$$

ADHM construction



fractional instanton

(see monopole constituent of 4d instanton)

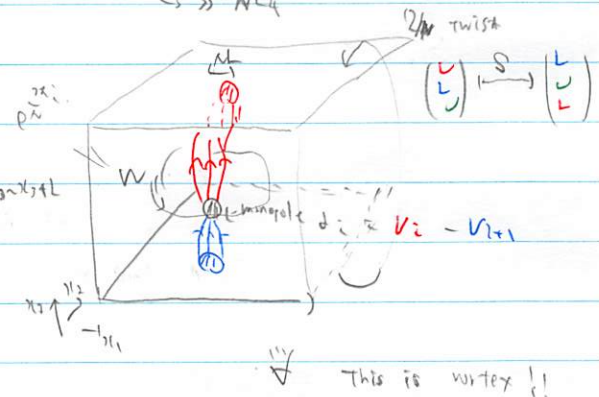
(Lee '96, '98, Krum, van Pelt '99)

$$\frac{1}{N} \ln(W) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left(e^{2\pi i n \cdot \frac{1}{N}} \right) \cdot \frac{1}{N} \text{Tr} \left(e^{2\pi i n \cdot \frac{1}{N}} \right)$$



$\Rightarrow$ center vortex

$$Q_{top}(\omega) = \frac{1}{N}$$



$$S = \frac{1}{g^2} \int d^4x (F \wedge F) \approx \frac{8\pi^2}{g^2} \left| \frac{Q_{top}}{N} \right|$$

② Dilute vortex-antivortex gas approximation for $SU(N)$ YM on $R^2 \times T^2_{\text{twist}}$

Partition function:

$$Z(\theta) = \int \mathcal{D}a \exp \left(- \frac{1}{g^2} \int_{T^2} (da + f) + i \frac{\theta}{8\pi^2} \int_{T^2} (da + f)^2 \right)$$

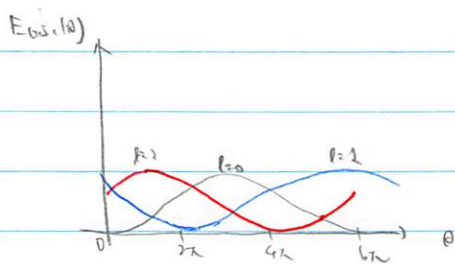
$$= \sum_{\text{vort.}} \left| \begin{array}{c} \text{vortex} \\ \text{anti-vortex} \end{array} \right| : \text{Vortex - anti-vortex gas}$$

$$\rightarrow \sum_{\substack{\# \text{ of vortex} \\ \# \text{ of anti-vortex}}} \left(\text{Vol} \times \# e^{-\frac{f^2}{2N}} \times \frac{\theta}{N} \right)^n \left(\text{Vol} \times \# e^{-\frac{f^2}{2N} - \frac{\theta}{N}} \right)^{\bar{n}}$$

Vol = volume of T^2

$$= \sum_{l=0}^{N-1} \exp \left[- \text{Vol} \times \left(-2 \# e^{-\frac{f^2}{2N}} \cos \left(\frac{\theta + 2\pi l}{N} \right) \right) \right]$$

consistent w/ anomaly
~ matching



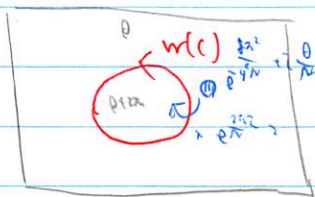
→ We have N branch structure for the θ -vacuum

③ This is a big difference compared with
an old-fashioned dilute instanton gas

$$E_{\text{inst}}(\theta) \sim e^{-\frac{8\pi^2}{g^2}} \cos(\theta)$$

this is ruled out
for confinement vacuum
via anomaly anomaly
- matching

• Area law for $W(C)$ in R^2



again $\frac{1}{N}$ factor of the vortex goes inside $W(C)$

$$\langle W(C) \rangle_{\theta \in 2\pi\mathbb{Z}} \approx e^{-\text{Area}(C) \times (E_0(\theta + 2\pi) - E_0(\theta))}$$

$$E_0(\theta + 2\pi) - E_0(\theta) = \sigma(\theta) : \text{string tension}$$

⇒ 2d Wilson loop is also confined!

• Semi-classical analysis suggests

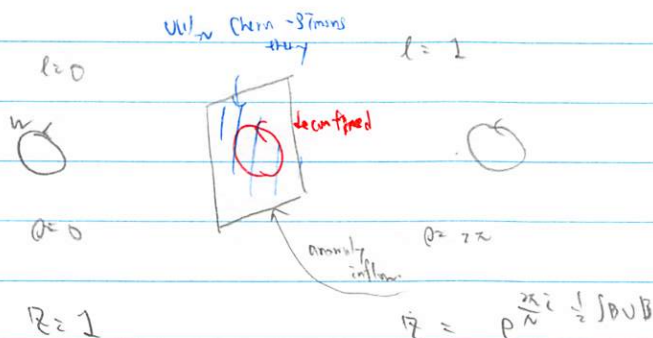
there are N confinement vacua (labeled by $l=0, 1, \dots, N-1$)

$$\langle W(C) \rangle_{\theta \in 2\pi\mathbb{Z}} = e^{-\text{Area}(C) (E_{0l} - E_l)} \xrightarrow{\sim 0} \text{Area-law}$$

What discriminates these vacua?

- For l -th vacuum, $H(\phi, \bar{\phi})|_{\phi=0} = \text{residue law}$
 $\Rightarrow (m, e) = (1, l)$ dyon condensing vacuum.
- $\mathcal{Z}(B) = e^{\frac{2\pi i}{N} \frac{1}{2} \int B \wedge B}$ is l -th SPT phase.

Gaiotto, Kapustin, Komargodski, Seiberg '17.



Application to QCD w/ fundamental quarks.

Can we take a twisted B.C. for QCD?
 = twisted B.C. using center sym.

Nahly, the answer should be No.

because the center sym. is explicitly broken w/ fundamental quarks.

$\Rightarrow$ We need some clever trick to implement it.

UV Dyon background magnetic flux.

	$SU(N)$ gauge	UV Dyon
γ	$\square$	$\frac{1}{N}$
$\bar{\gamma}$	$\bar{\square}$	$-\frac{1}{N}$

We introduce A_B s.t.

$$\frac{1}{2\pi} \int_{T^2} dA_B = m \in \mathbb{Z}$$

subtly field.

$$D_\mu \psi = \left[\partial_\mu + i \left(A_\mu^f + \frac{1}{N} A_{B,\mu} \right) \right] \psi$$

$$\psi(x_1, x_4) = g_3^{-1}(x_4) e^{-i \frac{1}{N} \int_3^4 A_B} \psi(0, x_4) \quad (1)$$

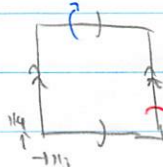
$$\psi(x_1, L) = g_4^{-1}(x_1) e^{-i \frac{1}{N} \int_1^L A_B} \psi(x_1, 0) \quad (2)$$

$$A_B(x_3 + L) = A_B(x_3) + \frac{2\pi}{L} dx_4$$

$$= i e^{-i \frac{1}{N} \int_3^4 A_B} \int_3^4 A_B$$

$$\left(A_B = \frac{2\pi x_3}{L^2} dx_4 \right)$$

$$(x_3, x_4), e^{i \frac{1}{N} \int_3^4 A_B}$$



$$(g_3(x_4), e^{i \frac{1}{N} \int_3^4 A_B})$$

no cycle cond:

$$\gamma(L, 1) \xrightarrow{(2)} \gamma(L, 0)$$

0 ↓

$$\gamma(0, L) \xrightarrow{(2)} \gamma(0, 0)$$

$$\Rightarrow g_3^{-1}(L) e^{i \frac{1}{\Lambda} \phi_3(L)} \cdot g_4^{-1}(0) e^{-i \frac{1}{\Lambda} \phi_4(0)}$$

$$= g_4^{-1}(L) e^{-i \frac{1}{\Lambda} \phi_4(L)} \cdot g_3^{-1}(0) e^{-i \frac{1}{\Lambda} \phi_3(0)}$$

$$\frac{1}{m} = \frac{1}{m} \int dA \cdot = \frac{1}{m} [(\phi_3(L) - \phi_3(0)) - (\phi_4(L) - \phi_4(0))]$$

⇒ By substituting this result into the no cycle condition.

we have

$$g_4(0) g_3(L) = g_3(0) g_4(L) e^{i \frac{1}{\Lambda} (\phi_3(L) - \phi_3(0) - \phi_4(L) + \phi_4(0))}$$

$e^{i \frac{1}{\Lambda} (\phi_3(L) - \phi_4(L))} = 1$

(up to gauge transformation)

$$\begin{cases} g_3(1/L) = S \\ g_4(1/L) = C \end{cases}$$

At Higgs twisted B.C.

for the gauge sector

Underlying reason for this mechanism:

Global structure of gauge & global sym.

$$SU(N)_{\text{gauge}} \times U(1)_{\text{global}} \rightarrow U(1)$$

$U(1)_{\text{global}} \rightarrow U(1)_B$

$$\frac{1}{2\pi} \int dA_\mu = 1$$

$$\Leftrightarrow \begin{cases} \frac{1}{m} \int dA_\mu = \frac{1}{\Lambda} \\ \int B_C = \frac{2\pi}{\Lambda} \end{cases}$$

↑
color - center flux

while the quantization of flux is violated.

$$\text{QCD} : R^2 \times T^2 \text{ w/ } \int_{T^2} dA_B = 2\pi$$

⇒ semi-classical set up

$$\begin{cases} \text{gluon} = \frac{2\pi}{\Lambda L} \text{ gapped and gas of center vortices} \\ \text{quark} = \text{4d Dirac fermion} \end{cases}$$

T^2 reduction ⇒ 2d massless Dirac fermion

(for each flavor)

$$\text{Index} \left(\not{D} \cdot T^2 \text{ w/ } A_B \right) = \frac{1}{2\pi} \int dA_B = 1$$

⇒ 1-flavor 2d Dirac massless mode

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2d Dirac fermion + $\frac{1}{N}$ = Abelian vortices
= 2d Schwinger model + charge N Abelian-Higgs model

→ We can solve this model conveniently
using 2d Bose-Fermi duality.

$$\bar{\psi}_L \psi_R \leftrightarrow e^{i\eta'}$$

Boson for describing the phase of chiral condensate

(For multi-flavor case,
it's good use to use
non-Abelian bosonization)

Let's derive the effective potential for η'

① Springs U(1)A sym.

$$\begin{cases} \psi \rightarrow e^{i\eta'_3} \psi \\ \bar{\psi} \rightarrow \bar{\psi} e^{-i\eta'_3} \end{cases} \quad \begin{aligned} & \bar{\psi} \psi \\ & \rightarrow e^{i\eta'_3} \frac{1}{m^2} \int d\eta' \delta(\eta') \times \bar{\psi} \psi \end{aligned}$$

(Fujikawa)

If we combine U(1)A w/ $\theta \rightarrow \theta - 2\pi$,

the QCD partition function is invariant.
(gauge sym.)

$$e^{i\eta'} \leftrightarrow \bar{\psi}_L \psi_R$$

$$(\bar{\psi}_L \psi_R \xrightarrow{U(1)_A} e^{i\eta'} \bar{\psi}_L \psi_R)$$

$$\Rightarrow U(1)_A : \eta' \mapsto \eta' + 2\pi$$

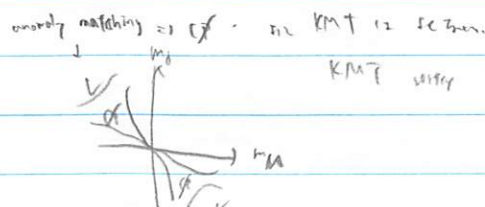
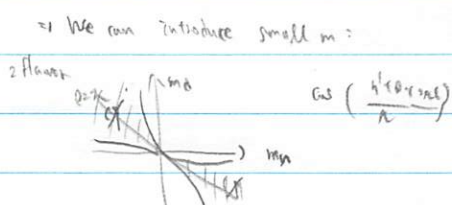
$$\mathcal{Z}_{\text{QCD on } \mathbb{R}^2 \times T^2 \text{ w/ Baryon flux}} = \int d\eta' e^{-\int \frac{1}{m^2} (\partial \eta')^2} \times \sum_{\ell=0}^{N-1} \exp \left(- \int d^2x \frac{1}{2} \left(\partial \eta' + \frac{2\pi \ell}{N} \right)^2 \right) \cos \left(\frac{\eta' + 2\pi m \ell}{N} \right)$$

dilute vortex gas.

($\Leftarrow$ This is different from the Kobayashi-Maskawa-Higgs vertex
which gives $\sim \cos(\eta' + \theta)$)

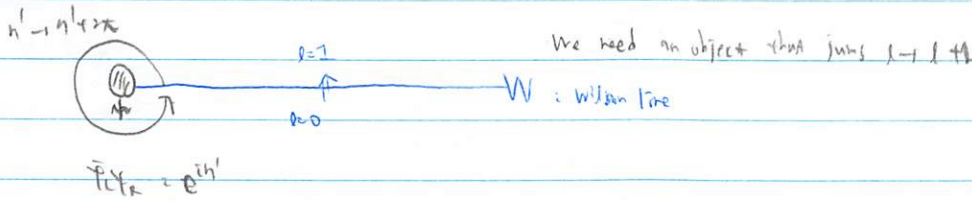
$\Rightarrow$ Dynamical variable for EFT: (η', ℓ)

U(1)A phase $\leftrightarrow$ discrete variable
 $\eta' \mapsto \eta' + 2\pi$



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$\eta' = \text{vortex}$



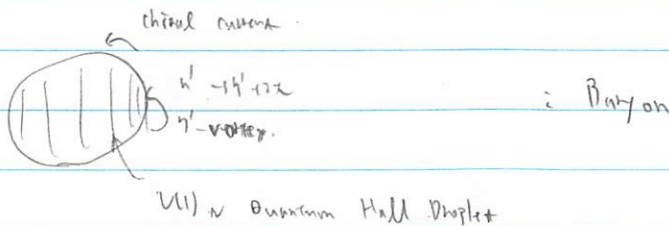
$\Rightarrow$ In 2d setup, η' -vortex = quark of itself

4d interpretation

Komar-Goldski's pancake baryon

$$\left(\begin{array}{l} \text{For } N_c \geq 2 \text{ QCD} : SU(N_c) \text{ chiral Lag.} \\ \rightarrow T_3(SU(N_c)) \approx \frac{1}{2} : \text{Skyrmion} = \text{Baryon} \\ \text{However, for } N_c = 1, \text{ so such Skyrmion exists.} \end{array} \right)$$

(In large- N_c)



$U(1) \sim$ Quantum Hall Droplet

By exciting the chiral current using vertex op.

we have $J = \frac{N_c}{2}$ state

$$\begin{aligned} & \text{U(1)-CS } \frac{1}{4\pi} \text{ Baryon} \\ & \int_{C \times T^2} \frac{N_c}{4\pi} a da \rightarrow \frac{2\pi}{\pi} \text{ mag. flux for } T^2 \text{ is:} \\ & \downarrow T^2\text{-reduction} \quad = \int_C a \end{aligned}$$

η' -vortex So, our picture of 2d η' -vortex w/ $w(c)$ naturally appears from this pancake picture.